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Motion in a Periodic Potential Driven by Rectangular Pulses

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Exact solutions are found for the averaged velocity of an overdamped pendulum driven by a series of square pulses. The formalism developed can be applied to either periodic or random switching as well as cases intermediate between the two. The phenomenon known as phase locking can be shown to exist when periodic switching is used.

KEY WORDS: Nonlinear dynamics; Josephson junctions; stochastic processes.

1. INTRODUCTION

The equation that describes the dynamics of an overdamped driven pendulum,

$$\dot{\varphi} = a - \sin \varphi \equiv g(\varphi) \tag{1}$$

occurs in a number of different applications. These are exemplified by the resistively shunted Josephson junction,[(1)] the theory of charge density waves,[(2)] phase locking in electric circuits,[(3)] and mode locking in ring laser gyroscopes.[(4)] More recent applications have included the theory of the motion of fluxons in superconductors,[(5)] the motion of defects in convective fluids,[(6)] and the penetration of biological channels by ions.[3] It is easy to determine properties of the solution to Eq. (1) since the equation is solvable in closed form. The most significant of these is the basis of the

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[3] Molecular dynamics simulations[(7)] suggest that Eq. (1) is a first approximation in this case.

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remarkable voltage–current characteristics of the Josephson junction. If we define a (dimensionless) frequency ω by $\omega = (a^2 - 1)^{1/2}$, we readily find that

$$\langle \dot{\varphi} \rangle \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T \dot{\varphi}(\tau)\, d\tau = \begin{cases} 0 & \text{for} \quad a < 1 \\ \omega & \text{for} \quad a > 1 \end{cases} \tag{2}$$

The content of this equation is best understood in terms of the behavior of a pendulum. When the parameter a is small the pendulum can only perform small oscillations around its equilibrium point, while if a is sufficiently large the pendulum is able to execute complete rotations.

Equation (1) describes the dynamical behavior of an autonomous system. Quite often φ is also subject to external forces. These may be of two generic varieties, deterministic and stochastic, i.e., random noise, with the result that the equation for $\varphi(\tau)$ can be expressed as

$$\dot{\varphi} = g(\varphi) + f_{\text{det}}(t) + f_{\text{stoch}}(t) \tag{3}$$

The deterministic forces whose effects have been analyzed in the context of dynamical systems are generally taken to be periodic, with applications to an external ac current driving the system, while the stochastic component can generally be identified with thermal noise.

While one can find a solution in closed form to Eq. (1), this is not true of Eq. (3) except in special cases. For example, when $f_{\text{stoch}}(t)$ is Gaussian white noise and $f_{\text{det}}(t) = 0$ the effect of the noise on the stationary voltage–current characteristics has been determined.[1] In a recent calculation we have determined the solution to Eq. (3) when $f_{\text{stoch}}(t) = 0$ and $f_{\text{det}}(t)$ is a periodic telegraph signal which may or may not be symmetric.[8] A number of authors have analyzed the case in which $f_{\text{det}}(t) = A \sin(\omega t)$ and $f_{\text{stoch}}(t) = 0$ either using perturbation theory[9] or by a numerical solution of the fundamental equation.[10] A most remarkable result of either calculation is the lock-in phenomenon which manifests itself as horizontal "Shapiro steps" that appear in the voltage–current characteristics of Josephson junctions. This phenomenon, which appears in the model specified by $f_{\text{det}}(t) = A \sin(\omega t)$ when $\langle \dot{\varphi} \rangle = n\omega$, $n = 1, 2, 3,\ldots$, is due to the stability of periodic orbits in phase space with respect to small changes in the parameter a that appears in the function $g(\varphi)$ in Eq. (3).

In the present paper we will discuss a number of exact solutions to Eq. (3). These are of two sorts, the first being those in which $f_{\text{stoch}}(t) = 0$ and $f_{\text{det}}(t)$ is one of a variety of different types of telegraph signal. Here we are able to bypass the limitation to small amplitudes of the driving function, which is a frequently used basis for a perturbation analysis of the solution to Eq. (3). This analysis will be described in the following section. In Section 3 we derive the fundamental equations for dichotomous noise, which are then applied in Section 4 to the case in which $f_{\text{det}}(t)=0$ while $f_{\text{stoch}}(t)$ is a two-state semi-Markov process with constant amplitudes. In both of these cases the fact that regeneration points[11] can be identified greatly simplifies the analysis while not affecting the basic physical properties of the system. The final section contains a discussion of results.

2. THE DETERMINISTIC TELEGRAPH SIGNAL DRIVEN CASE

In this section we summarize the analysis of the case in which $f_{\text{det}}(t)$ is given by a telegraph signal which, in the most general case, is defined by

$$f_{\text{det}}(t)=\begin{cases} A & \text{for} \quad n(T_1+T_2)<t\leqslant(n+1)T_1+nT_2 \\ -B & \text{for} \quad (n+1)T_1+nT_2<t\leqslant(n+1)(T_1+T_2) \end{cases} \tag{4}$$

where $n=0, 1, 2,...$. By defining the deterministic signal in this way we are able to handle both the case of the symmetric signal ($T_1=T_2=T$; $A=B$) and the case of a train of delta-function pulses ($B=0$, $A\to\infty$, $T_1\to 0$ with $AT_1=1$). The principal idea, just as in the analysis of the Kronig–Penney model, is to match the properties of the solution at the time points $n(T_1+T_2)$ and at the remaining points at which this time is incremented by T_2 and by T_1+T_2. For simplicity we restrict ourselves to the case in which $A=B$ and in which $T_1=T_2=T/2$. Two frequencies naturally appear in the analysis of the problem. These are

$$\omega_1=[(a+A)^2-1]^{1/2}, \qquad \omega_2=[(a-A)^2-1]^{1/2} \tag{5}$$

We will assume that the parameters a and A have been chosen so that both frequencies in this last equation are real. It is also convenient, in expressing results of the mathematical analysis, to define a pair of parameters θ_i, $i=1$, 2, and to change the variable of interest from $\varphi(t)$ to $u(t)$ by introducing the transformations

$$u(t)=\tan\left(\frac{\varphi(t)}{2}\right), \qquad \theta_i=\tan\left(\frac{\omega_i T}{2}\right) \tag{6}$$

We can solve Eq. (3) with $f_{\text{det}}(t)$ given in Eq. (4) and $f_{\text{stoch}}(t)=0$ exactly in any interval of periodicity $[2nT, (2n+1)T]$ or $[(2n+1)T, 2(n+1)T]$ since our use of the telegraph signal is equivalent to redefining the parameter a as $a\pm A$ in Eq. (1). Define the variables $U_n=u(2nT)$ and $V_n=u((2n+1)T)$. Our eventual goal is to calculate a recurrence relation for the U_n, which is possible because of the simple form of the periodic term.

The general form of the solution to Eqs. (3) and (4) in the interval $[2nT, (2n+1)T]$ can be written in implicit form as

$$t = \frac{2}{\omega_1}\left\{\tan^{-1}\left[\frac{(a+A)\,u(t)-1}{\omega_1}\right]+C\right\} \tag{7}$$

where C is a constant of integration. This may be found by setting $t = 2nT$, which allows us to calculate the value of C as

$$C = n\omega_1 T - \tan^{-1}\left[\frac{(a+A)\,U_n - 1}{\omega_1}\right] \tag{8}$$

Our knowledge of the form of C now allows us to propagate the solution to the next end of the interval, which provides a relation between V_n and U_n of the form

$$(a+A)\,V_n - 1 = \omega_1 \frac{\omega_1\theta_1 - 1 + (a+A)\,U_n}{\omega_1 + \theta_1 - (a+A)\theta_1 U_n} \tag{9}$$

In a similar fashion we can find a relation between U_{n+1} and V_n. When V_n is eliminated between this set of relations one finds a relation between successive values of U_n which can be expressed in the form of a linear fractional transformation as

$$U_{n+1} = \frac{P + QU_n}{R - SU_n} \tag{10}$$

where P, Q, R, and S are constants. These are written in terms of the parameters

$$\eta_1 = \frac{\theta_1}{\omega_1}, \qquad \eta_2 = \frac{\theta_2}{\omega_2}, \qquad \beta_1 = \frac{(a+A)\theta_1}{\omega_1}, \qquad \beta_2 = \frac{(a-A)\theta_2}{\omega_2} \tag{11}$$

as

$$\begin{aligned} P &= \beta_2(1+\eta_1) + \beta_1(1-\eta_2), \qquad & Q &= (1-\eta_1)(1-\eta_2) - \beta_1\beta_2 \\ R &= (1+\eta_1)(1+\eta_2) - \beta_1\beta_2, \qquad & S &= \beta_2(1-\eta_1) + \beta_1(1+\eta_2) \end{aligned} \tag{12}$$

Equation (10) can be solved analytically by rescaling it to

$$X_{n+1} = \frac{AX_n + B}{X_n + 1}$$

and then defining a new variable y by

$$X_n = \frac{y_{n+1}}{y_n} - 1 \Rightarrow y \text{ satisfies } y_{n+2} - (1+A)\, y_{n+1} + (A-B)\, y_n = 0$$

The presence of phase locking (or Shapiro steps in terminology descriptive of Josephson junctions) in the system corresponds to the existence of a fixed point in the transformation in Eq. (10). That is to say, the existence of phase locking is, in mathematical terms, equivalent to the existence of a fixed point U which is equal to $\lim_{n \to \infty} U_n$. When such a limiting value exists it is readily calculable from the quadratic equation that follows from Eq. (10). Details of the calculation of the existence and magnitude of Shapiro steps are given in ref. 8. The point to be emphasized, however, is that an exact result has been obtained without the restrictions implied by a perturbation theory analysis. Similar results can be obtained for the case of a periodic train of delta functions as originally discussed by Azbel and Bak.[12]

3. FUNDAMENTAL EQUATIONS FOR DICHOTOMOUS NOISE

Let us next consider the second of the cases of interest, in which $f_{\mathrm{det}}(t) = 0$ and $f_{\mathrm{stoch}}(t)$ takes the form of a random dichotomous signal. The same strategy can be applied to develop a formalism to calculate the response of the system using the basic idea of regeneration points as was used in Section 2, except that these points will now be allowed to occur at random times. We will discuss a case in which $f_{\mathrm{stoch}}(t)$ is allowed to take on two values $+A$ and $-B$ ($A, B \geqslant 0$), with transitions between these two values occurring at random times $0 < t_1 < t_2 < t_3 < \cdots$. To define the system more precisely, we will assume that the random intervals $T_0 = t_1$, $T_n = t_n - t_{n-1}$, $n > 0$, are identically distributed independent random variables characterized by a probability density function denoted by $\psi(t)$. The probability that a given time interval is $\geqslant t$ will be denoted by $\Psi(t) = \int_t^\infty \psi(\tau)\, d\tau$.

To the degree of generality which we have used in defining the system, the evolution of $\varphi(t)$ can be modeled in terms of a semi-Markov process of a particularly simple kind, i.e., a two-state model in which the state at time t is defined by the value of $f_{\mathrm{stoch}}(t)$. Since $\varphi(t)$ must now be regarded as a random variable, we can only seek its properties in terms of a probability density $p(\varphi, t)$ for the event $\varphi(t) = \varphi$. Let $\varphi_+(t)$ be the solution to

$$\dot{\varphi}_+ = g(\varphi_+) + A \tag{13}$$

and let $\varphi_-(t)$ be the analogous function when $+A$ is replaced by $-B$. We will also assume, to keep our results simple, that $\varphi(0)$ is equally likely to start from either state, with probability 1/2, and that $t=0$ marks the beginning of a sojourn in the appropriate state, so that $\psi(t)$ is the probability density for t_1 as well as for the succeeding T's.

The first step in calculating $p(\varphi, t)$ will be to characterize the regeneration points. This will be done in terms of a pair of functions which we denote by $\eta_+(\varphi, t)$ and $\eta_-(\varphi, t)$. The function $\eta_+(\varphi, t)$ is the joint density for a sojourn in the + state to end at time t, the value of $\varphi(t)$ being φ at that time. This joint density is the solution to an integral equation which can be written as

$$\eta_+(\varphi, t)=\tfrac{1}{2}\psi(t)\,\delta(\varphi-\varphi_+(t)) \\ +\int_{-\infty}^{\infty} d\varphi' \int_0^t \eta_-(\varphi', \tau)\,\psi(t-\tau)\,\delta(\varphi-\varphi'-\varphi_+(t-\tau))\,d\tau \tag{14}$$

with an analogous relation for $\eta_-(\varphi, t)$, in which the subscripted signs are everywhere reversed. In this last equation the first term on the right-hand side accounts for the possibility that the stay in the + state was the first, and the second term accounts for all of the remaining possibilities. In order to find $p(\varphi, t)$, we decompose this function into a sum of two terms, $p(\varphi, t)=p_+(\varphi, t)+p_-(\varphi, t)$, the first term accounting for the system being in the + state and $\varphi(t)=\varphi$ and the second to the remaining case. The function $p_+(\varphi, t)$ then satisfies

$$p_+(\varphi, t)=\tfrac{1}{2}\Psi(t)\,\delta(\varphi-\varphi_+(t)) \\ +\int_{-\infty}^{\infty} d\varphi' \int_0^t \eta_-(\varphi', \tau)\,\Psi(t-\tau)\,\delta(\varphi-\varphi'-\varphi_+(t-\tau))\,d\tau \tag{15}$$

a similar equation being valid for $p_-(\varphi, t)$. When there is additional noise in the system, so that, for example, Eq. (13) is replaced by a Langevin equation, the delta-function terms in Eqs. (14) and (15) will be replaced by some more complicated propagator.

A formal solution to Eqs. (14) and (15) can always be derived in terms of a two-dimensional Laplace transform, but the only explicit solution for $p(\varphi, t)$ that we have managed to find corresponds to the Markovian case discussed in the monograph by Horsthemke and Lefever.[13] It is instructive to see how their equations can be found from our more general formalism. The Markovian case corresponds to choosing the waiting-time density to have the form $\psi(t)=\gamma e^{-\gamma t}$, from which one finds $\Psi(t)=\psi(t)/\gamma$. Since Eqs. (14) and (15) are then identical up to a factor of γ, it follows that

$$p_+(\varphi, t)=\eta_+(\varphi, t)/\gamma, \qquad p_-(\varphi, t)=\eta_-(\varphi, t)/\gamma \tag{16}$$

so that in this case we need only solve Eq. (14) together with its conjugate rather than dealing separately with Eqs. (14) and (15).

Let us express the solution of Eq. (13) for t in terms of φ_+ as

$$t_+(\varphi \mid \varphi_0) = \int_{\varphi_0}^{\varphi} \frac{dz}{g(z)+A} \tag{17}$$

where $\varphi_0 = \varphi(0)$. Observe now that the delta-function terms in Eqs. (14) and (15) can be replaced by

$$\delta(\varphi - \varphi_+(t)) = \frac{\delta(t - t_+(\varphi \mid \varphi_0))}{d\varphi/dt} = \frac{\delta(t - t_+(\varphi \mid \varphi_0))}{g(\varphi)+A} \tag{18}$$

After substituting this representation of the delta function for the first term on the right-hand side of Eq. (15), taking the Laplace transform $\hat{p}_+(\varphi, s)$ $[=\mathscr{L}_t\{p_+(\varphi, t)\}]$, and multiplying both sides of the equation by $g(\varphi)+A$, we are able to express the equation for $\hat{p}_+(\varphi, s)$ as

$$\begin{aligned}[g(\varphi)+A]\, \hat{p}_+(\varphi, s) &= \tfrac{1}{2} e^{-(\gamma+s)\, t_+(\varphi \mid ')} \\ &\quad + \gamma \int_{-\infty}^{\varphi} e^{-(\gamma+s)\, t_+(\varphi \mid \varphi')} \hat{p}_-(\varphi', s)\, d\varphi' \end{aligned} \tag{19}$$

Notice that we have replaced the upper limit on the φ' integral by φ, which is the maximum attainable value when the system is in the + state. Finally,

$$\frac{dt_+(\varphi \mid \varphi')}{d\varphi} = \frac{1}{g(\varphi)+A} \tag{20}$$

where it is important to observe that the right side of the equation is independent of φ'. We find that, on differentiating both sides of Eq. (19) with respect to φ and eliminating the integral term through the use of the same equation, $\hat{p}(\varphi, s)$ satisfies the partial differential equation

$$\begin{aligned}&\frac{\partial}{\partial \varphi} \{[g(\varphi)+A]\, \hat{p}_+(\varphi, s)\} \\ &\quad = -\frac{(\gamma+s)}{g(\varphi)+A} \{[g(\varphi)+A]\, \hat{p}_+(\varphi, s)\} + \gamma \hat{p}_-(\varphi, s)\end{aligned} \tag{21}$$

Since the parameter s appears in this relation in a very simple form, we may perform the inversion in closed form to arrive at an equation for $p_+(\varphi, t)$ which reads

$$\frac{\partial p_+}{\partial t} = -\frac{\partial}{\partial \varphi} \{[g(\varphi)+A]\, p_+\} + \gamma(p_- - p_+) \tag{22}$$

with a similar equation with the +'s and −'s interchanged. These are just the equations derived by Horsthemke and Lefever.[13]

4. THE JOSEPHSON JUNCTION WITH NOISE

Let us next consider the calculation of the effects of Markovian dichotomous noise on the dynamics of a Josephson junction. That is to say, we will solve the equation

$$\dot{\varphi} = g(\varphi) + n(t) \tag{23}$$

where $n(t)$ is a two-state Markov telegraph signal which is allowed to take on the values A and $-B$, where $A, B > 0$. The rate for the transition $A \to -B$ will be denoted by γ and the reverse rate will be denoted by γ'. We further fix the parameters to ensure that $\langle n(t) \rangle = 0$, which implies the relation

$$\gamma' A = \gamma B \tag{24}$$

We expect, as a result of the noise, that phase locking or Shapiro steps cannot appear because of the loss of coherence.

As in the last section, we define two probability densities $p_+(\varphi, t)$ and $p_-(\varphi, t)$ which correspond to the evolution of $\varphi(t)$ subject ot the values of the noise being A and $-B$, respectively, where again we need the restrictions $(a-B)^2$, $(a+A)^2 > 1$. The set of equations satisfied by these two functions is a slight generalization of that shown in Eq. (22), i.e.,

$$\begin{aligned} \frac{\partial p_+}{\partial t} &= -\frac{\partial}{\partial \varphi} \{[g(\varphi) + A] p_+\} + \gamma' p_- - \gamma p_+ \\ \frac{\partial p_-}{\partial t} &= -\frac{\partial}{\partial \varphi} \{[g(\varphi) - B] p_-\} + \gamma p_+ - \gamma' p_- \end{aligned} \tag{25}$$

It is useful to replace this set of equations by an equivalent set for the functions

$$p = p_+ + p_- \qquad \text{and} \qquad q = \gamma p_+ - \gamma' p_- \tag{26}$$

These functions are found from Eq. (25) to satisfy the equations

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial \varphi} \left\{ g(\varphi) p + \frac{A}{\gamma} q \right\} = -\frac{\partial J}{\partial \varphi} \tag{27a}$$

$$\frac{\partial q}{\partial t} = -\frac{\partial}{\partial \varphi} \left\{ \left[g(\varphi) + \frac{A}{\gamma} (\gamma - \gamma') \right] q + A\gamma' p \right\} - (\gamma + \gamma') q \tag{27b}$$

The variable J in Eq. (27a) is seen to be a flux, which is related to the fact that the function p is a density.

Let the stationary solutions to Eq. (25) be denoted by $p_{+}^{(\mathrm{st})}(\varphi)$ and $p_{-}^{(\mathrm{st})}(\varphi)$, respectively, and those of Eq. (27) by $p^{(\mathrm{st})}(\varphi)$ and $q^{(\mathrm{st})}(\varphi)$. Then, because the right-hand side of the dynamical equation depends on φ only as $\sin\varphi$, we can express $\langle\dot{\varphi}\rangle$ as

$$\begin{aligned}\langle\dot{\varphi}\rangle &= \int_{-\pi}^{\pi} \{[g(\varphi)+A]\, p_{+}^{(\mathrm{st})}(\varphi) + [g(\varphi)-B]\, p_{-}^{(\mathrm{st})}(\varphi)\}\, d\varphi \\ &= \int_{-\pi}^{\pi} \left\{ g(\varphi)\, p^{(\mathrm{st})}(\varphi) + \frac{A}{\gamma}\, q^{(\mathrm{st})}(\varphi) \right\} d\varphi \end{aligned} \tag{28}$$

The stationary state is characterized by the fact that the flux J is a constant. It therefore follows from Eq. (27a) that $p^{(\mathrm{st})}$ and $q^{(\mathrm{st})}$ are related by

$$g(\varphi)\, p^{(\mathrm{st})}(\varphi) + \frac{A}{\gamma}\, q^{(\mathrm{st})}(\varphi) = J \tag{29}$$

so that $\langle\dot{\varphi}\rangle$ can be written in the simple form

$$\langle\dot{\varphi}\rangle = 2\pi J \tag{30}$$

Thus we see that a derivation of the form for $\langle\dot{\varphi}\rangle$ is equivalent to a calculation of the flux J.

The stationary solutions are determined by setting the time derivatives in Eq. (27) to 0. In doing so, we eliminate the term $q^{(\mathrm{st})}(\varphi)$ in terms of $p^{(\mathrm{st})}(\varphi)$ from Eq. (29) and substitute the result into Eq. (27b) with $\partial q/\partial t = 0$, thereby finding an equation containing only $p^{(\mathrm{st})}(\varphi)$. This equation will be written in terms of two functions $\Gamma(\varphi)$ and $\Omega(\varphi)$, which are defined in terms of the function $g(\varphi)$ and its derivative $g'(\varphi)$ as

$$\begin{aligned}\Gamma(\varphi) &= \frac{g'[2g+(A/\gamma)(\gamma-\gamma')]+(\gamma+\gamma')\, g}{g[g+(A/\gamma)(\gamma-\gamma')]-A^2\gamma'/\gamma} \\ \Omega(\varphi) &= \frac{g'+\gamma+\gamma'}{g[g+(A/\gamma)(\gamma-\gamma')]-A^2\gamma'/\gamma}\end{aligned} \tag{31}$$

as

$$\frac{dp^{(\mathrm{st})}}{d\varphi} + \Gamma(\varphi)\, p^{(\mathrm{st})} = J\Omega(\varphi) \tag{32}$$

This has a solution which can be written as

$$p^{(\mathrm{st})}(\varphi) = \frac{J}{Q(\varphi)} \int_{-\pi}^{\varphi} \Omega(\xi)\, Q(\xi)\, d\xi + \frac{C}{Q(\varphi)} = \frac{JI(\varphi)+C}{Q(\varphi)} \tag{33}$$

in which $Q(\varphi)$ is the function

$$Q(\varphi) = \exp\left(\int_{-\pi}^{\varphi} \Gamma(\xi)\, d\xi\right) \tag{34}$$

$I(\varphi)$ is the integral appearing in Eq. (33), and C is a constant of integration. There are two constants to be determined in the general solution in Eq. (33), and therefore two conditions are required to fix these constants. These are normalization and periodicity, which is to say that C and J can be found by imposing the conditions

$$p^{(\mathrm{st})}(\pi) = p^{(\mathrm{st})}(-\pi) \qquad \text{and} \qquad \int_{-\pi}^{\pi} p^{(\mathrm{st})}(\varphi)\, d\varphi = 1 \tag{35}$$

These constraints lead to an expression for J $(= \langle \dot{\varphi} \rangle / 2\pi)$ which is found to have the form

$$J = \left\{ \int_{-\pi}^{\pi} \frac{I(\xi) + K}{Q(\xi)}\, d\xi \right\}^{-1} \tag{36}$$

the constant K being

$$K = I(\pi)/[Q(\pi) - 1] \tag{37}$$

These relations are simplified to a considerable extent in the special case in which $\gamma = \gamma'$ so that $A = B$. When this condition is satisfied one finds

$$\Gamma(\varphi) = \frac{2g(\varphi)\, g'(\varphi) + 2\gamma g(\varphi)}{g^2(\varphi) - A^2}, \qquad \Omega(\varphi) = \frac{g(\varphi) + 2\gamma}{g^2(\varphi) - A^2} \tag{38}$$

The function $Q(\varphi)$ in Eq. (34) can then be expressed as

$$\begin{aligned} Q(\varphi) &= \frac{g^2(\varphi) - A^2}{a^2 - A^2} \exp\left\{ 2\gamma \int_{-\pi}^{\varphi} \frac{g(\varphi')}{g^2(\varphi') - A^2}\, d\varphi' \right\} \\ &= \frac{g^2(\varphi) - A^2}{a^2 - A^2} \exp[\gamma \Delta(\varphi)] \end{aligned} \tag{39}$$

in which

$$\Delta(\varphi) = \int_{-\pi}^{\varphi} \left(\frac{1}{g(\varphi') - A} + \frac{1}{g(\varphi') + A} \right) d\varphi' \tag{40}$$

Also, since $g(\pi) = a$, the value of $Q(\pi)$ is $\exp[\gamma \Delta(\pi)]$ and the function $I(\varphi)$ is

$$I(\varphi) = \frac{1}{a^2 - A^2} \int_{-\pi}^{\varphi} [2\gamma + g(\xi)] \exp\{\gamma \Delta(\xi)\}\, d\xi \tag{41}$$

The combination of all of these relations allows us to express the flux J as

$$J = [Q(\pi) - 1] \left\{ \int_{-\pi}^{\pi} \frac{\exp[-\gamma \Delta(\xi)]}{g^2(\xi) - A^2} F(\xi)\, d\xi \right\}^{-1} \tag{42}$$

with the function $F(\xi)$ defined by

$$F(\xi) = (a^2 - A^2)\, I(\pi) + [Q(\pi) - 1] \int_{-\pi}^{\xi} \{2\gamma + g(\rho)\} \exp[\gamma \Delta(\rho)]\, d\rho \tag{43}$$

Our formalism allows us to recover some results of Chen and Dong[14] for vortex diffusion in high-T_c superconductors which is based on the solution to Eq. (3) with $f_{\text{stoch}}(t)$ = white noise. The white noise limit in the present notation is defined by the limits

$$A \to \infty, \qquad \gamma \to \infty, \qquad \lim_{A, \gamma \to \infty} \frac{A^2}{\gamma} = 2\sigma^2 \tag{44}$$

where σ^2 is a constant proportional to kT. The limiting behavior in this last equation allows the simplification of some of the formulas appearing in the last paragraphs. For example, the constant $Q(\pi)$ becomes

$$Q(\pi) = \exp\left(-\frac{4\pi a \gamma}{A^2} \right) = \exp\left(-\frac{2\pi a}{\sigma^2} \right) \tag{45}$$

Some further simplifications in the algebra allow us to then write for the flux

$$J = 2\sigma^2 \left[1 - \exp\left(-\frac{2\pi a}{\sigma^2} \right) \right] \times \left\{ \int_{-\pi}^{\pi} \exp\left[\int_{-\pi}^{\xi} g(\rho)\, d\rho \right] d\xi \int_{\xi}^{\xi + 2\pi} \exp\left[\int_{-\pi}^{\zeta} g(\alpha)\, d\alpha \right] d\zeta \right\}^{-1} \tag{46}$$

which is equivalent to Eq. (10) of Chen and Dong.[14] Numerical calculations of the flux based on Eq. (42) for dichotomous noise suggest that it is always a monotonic function of the parameter a with a notable decrease as $a - A$ approaches the critical value of 1. This agrees qualitatively with the results of Chen and Dong for white noise.

5. DISCUSSION

We have considered the dynamical behavior of an overdamped pendulum subject to an external driving field which was taken to have the form of a series of rectangular pulses allowed to take on two values only. The switching times between the two values were of two kinds: periodic, as treated in Section 2, and random, as in Section 4. In both of these, the existence of regeneration points greatly simplifies the mathematical analysis while not doing violence to the physics of the system. As might have been anticipated, phase locking or Shapiro steps occur under the influence of a periodic driving field, and are destroyed, either partially or totally, when the driving field is random. The formalism developed in Section 3 permits an analysis of the transitions between these two cases.

Indeed, the choice of an exponential form for $\psi(t)$ reduces the problem of finding $p^{(\mathrm{st})}(\varphi)$, originally formulated in terms of the integral equations in Eqs. (14) and (15), to that of solving a pair of first-order differential equations which have a relatively simple form. It is relatively simple to show that, for example, by setting $\psi(t) = t\exp(-t)$, the integral equations are equivalent to a pair of second-order differential equations. The extension of these results to gamma distributions of integer order is straightforward, leading to systems of correspondingly higher-order differential equations. While these can generally be solved, the effort to do so increases considerably with the order of the gamma distributions. It would therefore be worthwhile to return to Eqs. (14) and (15) to solve them at least in terms of integral transforms. This is easy to do in principle but leads to some extremely complicated expressions which we have not so far been able to handle in any practical way. A physical question which motivates our wanting to make this further step is the following: We have shown in Section 2 that when $\psi(t)$ is a delta function there is phase locking in the solution to Eq. (3), while when $\psi(t)$ is an exponential function of time, phase locking is absent. Suppose that one considers a sequence of sojourn time densities $\psi_\lambda(t)$ with the property that for all values of λ the average time

$$\langle t(\lambda)\rangle \equiv \int_0^\infty t\psi_\lambda(t)\,dt = T \tag{47}$$

is fixed, and further that $\lim_{\lambda\to\infty}\psi_\lambda(t) = \delta(t-T)$. We can then ask whether there is a value of λ which induces the transition no phase locking $\to$ phase locking, or whether phase locking occurs only for $\lambda = \infty$. Such a family of densities is exemplified by

$$\psi_\lambda(t) = \frac{(\lambda+1)^{\lambda+1}\,t^\lambda}{T^{\lambda+1}\Gamma(\lambda+1)}\exp\left[-\frac{(\lambda+1)t}{T}\right] \tag{48}$$

This question remains open for future investigation.

REFERENCES

1. A. Barone and G. Paterno, *Physics and Applications of the Josephson Effect* (Wiley, New York, 1982).
2. G. Grüner, A. Zawadowski, and P. M. Chaikin, *Phys. Rev. Lett.* **46**:511 (1981).
3. A. J. Viterbi, *Principles of Coherent Communications* (McGraw-Hill, New York, 1966).
4. W. Schleich, C. S. Cha, and J. D. Cresser, *Phys. Rev. A* **29**:230 (1984).
5. B. Shapiro, I. Dayan, M. Gitterman, and G. H. Weiss, *Phys. Rev. B* **46**:8416 (1992).
6. V. Steinberg, Private communication.
7. B. Roux and M. Karplus, *Biophys. J.* **59**:961 (1991).
8. B. Shapiro, I. Dayan, M. Gitterman, and G. H. Weiss, *Phys. Rev. B* **46**:8349 (1992).
9. L. G. Aslamazov and A. I. Larkin, *JETP Lett.* **9**:150 (1968).
10. P. Russer, *J. Appl. Phys.* **43**:2008 (1972).
11. W. L. Smith, *Proc. R. Soc. A* **232**:6 (1955).
12. M. Ya. Azbel and P. Bak, *Phys. Rev. B* **30**:3722 (1984).
13. W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer-Verlag, Berlin, 1984).
14. B. Chen and J. Dong, *Phys. Rev. B* **44**:10206 (1991).